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RAPID CALCULATION OF TEMPERATURE IN A REGENERATIVE HEAT EXCHANGER HAVING ARBITRARY INITIAL SOLID AND ENTERING FLUID TEMPERATURES*

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Abstract—By means of simple hand calculations it is possible to determine fluid and solid temperatures at any time and location in the regenerative heat exchanger. The methods described apply to cases where the initial temperature varies arbitrarily with longitudinal position in the matrix and the entering fluid temperature varies arbitrarily with time. The solution to the problem with uniform initial matrix temperature and constant entering fluid temperature has been published previously [1–4] and is presented herein in the form of tables and curves for values of the parameters η and ξ from 0 to 20. The solutions for the linear initial matrix temperature and linear entering fluid temperatures are also presented in the form of tables and curves for the same range of parameters. By superposition, these results are extended to the cases of arbitrary initial matrix temperature and arbitrary entering fluid temperature. Either of two methods is useful in obtaining numerical results. One is to evaluate a convolution integral which involves the arbitrary condition. The other is to approximate the arbitrary initial (and/or boundary) condition by a number of linear segments and to superimpose the tabulated solutions.

NOMENCLATURE

A,	open flow area [ft^2];
C,	specific heat, wall [Btu/lb degR];
I_0, I_1,	modified Bessel functions of the first kind;
M,	mass per unit length, wall [lb/ft];
P,	wetted perimeter [ft];
R,	gas constant [ft lb$_f$/lb degR];
a_1, a_2,	derivatives of approximating function v_a;
b_1, b_2,	derivatives of approximating function u_a;
c_p, c_v,	specific heats, fluid (constant pressure, volume);
h,	heat-transfer coefficient [Btu/h ft^{2}degR];
$\dot{m}$,	fluid mass flow [lb/h];
p,	pressure [lb/ft^2];
q,	heat flux [Btu/h ft^2];
t,	fluid temperature °R];
t_w,	wall temperature;
u,	entering fluid temperature;
u_η,	entering fluid temperature at time η;
v,	initial wall temperature;
v_ξ,	initial wall temperature at location ξ;
x, y,	illustrative variables;
z,	distance from entrance [ft];
γ,	ratio of specific heats;
ζ,	dummy variable, $0 \leqslant \zeta \leqslant \xi$;
η,	$= hP\theta/MC$, dimensionless time;
θ,	time [h];
ξ,	$= hPz/\dot{m}c_p$, dimensionless length;
ρ,	fluid density [lb/ft^3];
τ,	dummy variable, $0 \leqslant \tau \leqslant \eta$.

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INTRODUCTION

A LARGE quantity of fluid can be quickly heated (or cooled) by passing it through a heat storage device. Much higher effective heat-transfer rates can be achieved during a short high mass flow period in such a heat exchanger than might be available from the primary heat source (or sink). As an example, consider the heating of a stream of air for a blowdown-type wind tunnel. Here, the mass flow rate may be of the order of 10 lb/s and the duration 1 min. Now, the time available for recharging the heat exchanger may be of the order of several hours. It therefore is most convenient to design the recharging system such that a low heat-transfer rate could be used over this entire period to replace the energy removed.

This type of operation is not strictly "periodic" since consecutive wind tunnel runs will require different mass flows, temperatures and durations, and recharging periods will vary as well. The approach to calculating the performance of such a heat exchanger has consequently been to determine the transient temperatures once the initial temperature distribution is given. Here, the initial temperature is far from uniform and the entering fluid temperature can vary appreciably with time. It is this application which has prompted the present investigation.

In 1926, Anzelius [1] obtained the solution for the condition of a constant entering fluid temtemperature. Solutions were also obtained by Nusselt [2], Hausen [3, 5], and Schumann [4] at about the same time. Jakob [6] presents a rather detailed account of these works. The plots given by Hausen for the case where the initial temperature is uniform have been reproduced by Jakob [6], and Eckert and Drake [7]. The present paper also includes such plots for use with the methods to be described.

A graphical method due to Hausen [8] which can treat also the arbitrary initial and boundary condition problems is described by Jakob. This is essentially a step-by-step procedure which requires iteration. The author has used a numerical adaptation of this method, which eliminates the need for iteration, in preparing Tables 1, 2 and 3 and Figs. 1–3. Another finite difference approach is the "heat-pole method" due to Hausen [8]. It deals with the problem of arbitrary initial temperature distribution, using superposition of heat-pole functions to construct the solution required.

A paper of Klinkenberg and Harmens [9] utilizes the solution to the problem with uniform initial temperature and constant entering fluid temperature as a fundamental function and develops the solution for arbitrary conditions. His results are in agreement with equations (7) and (10) derived in the analysis below.

ANALYSIS

For the purposes of analysis, we shall adopt the concept of the regenerative heater matrix being comprised of one or more identical, parallel, thin-walled ducts. It will be assumed that the wall is sufficiently thin and its thermal conductivity is sufficiently high such that temperature variation within the wall thickness is negligible. It will also be assumed that effects of conduction in both the wall and the fluid in the direction parallel to flow are negligible.

The heat transfer between the wall and the fluid is governed by the constant film coefficient, h. The heat-transfer area per unit length of the matrix—the wetted perimeter—is P. The mass per unit length of matrix material is M. These quantities, as well as specific heats, are constants.

In the matrix, consider an element of length Δz parallel to the direction of flow. See Fig. 4. Now at time θ, fluid enters the element at the temperature t_1 and leaves the element at the temperature t_2.

The internal energy in the control volume is $c_v t \rho A \Delta_z$. By the conservation of energy principle. the rate of internal energy change is equated to the net rate of energy passing into the control volume.

$$c_v \rho \frac{\partial t}{\partial \theta} A \Delta_z + c_v \frac{\partial \rho}{\partial \theta} t A \Delta_z = c_p \dot{m}_1 t_1 - c_p \dot{m}_2 t_2 + q P \Delta_z.$$

Rearranging and taking the limit as Δ_z tends to zero gives

$$c_v\rho\frac{\partial t}{\partial\theta}A + c_v\frac{\partial\rho}{\partial\theta}tA + c_p t\frac{\partial\dot{m}}{\partial z} + c_p\dot{m}\frac{\partial t}{\partial z} = qP. \tag{1}$$

From continuity and the assumption of a perfect gas there are the relationships

$$\frac{\partial\dot{m}}{\partial z} = -A\frac{\partial\rho}{\partial\theta}$$

and

$$\frac{\partial\rho}{\partial\theta} = \frac{1}{Rt}\frac{\partial p}{\partial\theta} - \frac{p}{Rt^2}\frac{\partial t}{\partial\theta}.$$

The convective heat transfer per unit length is $qP = hP(t_w - t)$. Making these substitutions, equation (1) becomes

$$\frac{A\rho}{\dot{m}}\frac{\partial t}{\partial\theta} - \left(1 - \frac{1}{\gamma}\right)\frac{A}{\dot{m}R}\frac{\partial p}{\partial\theta} + \frac{\partial t}{\partial z} = \frac{hP}{c_p\dot{m}}(t_w - t).$$

Normally the second term on the left-hand side is very small and may be neglected. In this case, we have equation (2),

$$\frac{A\rho}{\dot{m}}\frac{\partial t}{\partial\theta} + \frac{\partial t}{\partial z} = \frac{hP}{c_p\dot{m}}(t_w - t). \tag{2}$$

Secondly, the equation which governs the temperature of the stationary wall can be immediately written

$$\frac{\partial t_w}{\partial\theta} = \frac{hP}{MC}(t - t_w). \tag{3}$$

It was this mathematical model which Schumann analyzed [4].

Investigation of the orders of magnitude of the terms in equation (2) has shown that the term $(A\rho/\dot{m})\,\partial t/\partial\theta$ is usually small. Suppose, for example, the approximation

$$\frac{\partial t}{\partial\theta} \approx \frac{\partial t_w}{\partial\theta}.$$

This substitution with equations (2) and (3) gives

$$\frac{\partial t}{\partial z} \approx \frac{hP}{c_p\dot{m}}\left(1 + \frac{A\rho c_p}{MC}\right)(t_w - t). \tag{4}$$

Subtracting equation (4) from equation (2) yields

$$\frac{A\rho}{\dot{m}}\frac{\partial t}{\partial\theta} \approx -\frac{hP}{c_p\dot{m}}\frac{A\rho c_p}{MC}(t_w - t).$$

The ratio of the two terms of the left-hand side of equation (2) is now found to be

$$\frac{(A\rho/\dot{m})\,\partial t/\partial\theta}{\partial t/\partial z} \approx -\frac{1}{1 + MC/A\rho c_p}.$$

It is therefore apparent that when the ratio $MC/A\rho c_p$ is large, the term $(A\rho/\dot{m})\,\partial t/\partial\theta$ may be neglected.

The remainder of the analysis will be restricted to the simplified model having the following partial differential equations in dimensionless form, which come from equations (2) and (3), respectively. Assuming $(A\rho/\dot{m})\partial t/\partial\theta = 0$,

$$\frac{\partial T}{\partial\xi} = T_w - T \tag{5}$$

$$\frac{\partial T_w}{\partial\eta} = T - T_w \tag{6}$$

where $\xi = hPz/c_p\dot{m}$, $\eta = hP\theta/CM$
$T = (t - u)/(v - u)$, $T_w = (t_w - u)/(v - u)$.
u = entering fluid temperature
and v = initial wall temperature.

With the initial condition $T_w = 1$ at $\eta = 0$ and the boundary condition $T = 0$ at $\xi = 0$, Hausen [3] produced curves of the solution. He plotted the reduced temperatures against ξ with η as a parameter. Following this convention, Fig. 1 presents the solution for use in the current work. A tabular presentation is also supplied in Table 1. The values contained herein have been generated with a finite difference technique by digital computer. Comparison of these results with the exact solution (equation 24) has been made at several points. This comparison and details regarding the finite difference technique are included in the Appendix.

Examination of equations (5) and (6) and the initial and boundary conditions discloses a sort of symmetry. Consequently, the temperature

Table 1. Reduced temperature function $T(\eta, \xi)$, $1 - T_w(\eta_w, \xi_w)$

η, ξ_w	$\xi, \eta_w = 0$	1	2	3	4	5	6	7	8	9	10
0	0·0000	0·6321	0·8647	0·9502	0·9817	0·9933	0·9975	0·9991	0·9997	0·9999	1·0000
1	0·0000	0·3456	0·6058	0·7752	0·8768	0·9345	0·9660	0·9827	0·9913	0·9957	0·9979
2	0·0000	0·1824	0·3964	0·5853	0·7301	0·8315	0·8984	0·9404	0·9659	0·9809	0·9895
3	0·0000	0·0937	0·2468	0·4166	0·5731	0·7019	0·7998	0·8698	0·9178	0·9493	0·9694
4	0·0000	0·0472	0·1479	0·2830	0·4282	0·5650	0·6821	0·7757	0·8466	0·8978	0·9336
5	0·0000	0·0233	0·0860	0·1850	0·3069	0·4360	0·5590	0·6672	0·7567	0·8271	0·8803
6	0·0000	0·0114	0·0487	0·1171	0·2123	0·3244	0·4418	0·5544	0·6555	0·7412	0·8106
7	0·0000	0·0055	0·0270	0·0722	0·1424	0·2336	0·3378	0·4462	0·5508	0·6459	0·7282
8	0·0000	0·0026	0·0147	0·0434	0·0930	0·1635	0·2508	0·3486	0·4497	0·5478	0·6379
9	0·0000	0·0012	0·0079	0·0256	0·0594	0·1115	0·1813	0·2650	0·3574	0·4526	0·5453
10	0·0000	0·0006	0·0042	0·0148	0·0371	0·0744	0·1279	0·1966	0·2771	0·3649	0·4551
11	0·0000	0·0003	0·0022	0·0085	0·0228	0·0486	0·0883	0·1425	0·2098	0·2874	0·3713
12	0·0000	0·0001	0·0011	0·0048	0·0137	0·0311	0·0597	0·1011	0·1556	0·2216	0·2965
13	0·0000	0·0001	0·0006	0·0026	0·0082	0·0196	0·0397	0·0704	0·1130	0·1673	0·2320
14	0·0000	0·0000	0·0003	0·0014	0·0048	0·0122	0·0259	0·0482	0·0806	0·1240	0·1780
15	0·0000	0·0000	0·0002	0·0008	0·0028	0·0074	0·0167	0·0324	0·0565	0·0903	0·1342
16	0·0000	0·0000	0·0001	0·0004	0·0016	0·0045	0·0106	0·0215	0·0390	0·0646	0·0994
17	0·0000	0·0000	0·0000	0·0002	0·0009	0·0027	0·0066	0·0140	0·0265	0·0456	0·0725
18	0·0000	0·0000	0·0000	0·0001	0·0005	0·0016	0·0041	0·0090	0·0178	0·0317	0·0521
19	0·0000	0·0000	0·0000	0·0001	0·0003	0·0009	0·0025	0·0058	0·0118	0·0217	0·0369
20	0·0000	0·0000	0·0000	0·0000	0·0002	0·0005	0·0015	0·0036	0·0077	0·0147	0·0258
η, ξ_w	$\xi, \eta_w = 10$	11	12	13	14	15	16	17	18	19	20
0	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
1	0·9979	0·9990	0·9995	0·9998	0·9999	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
2	0·9895	0·9943	0·9970	0·9984	0·9992	0·9996	0·9998	0·9999	1·0000	1·0000	1·0000
3	0·9694	0·9819	0·9895	0·9940	0·9966	0·9981	0·9990	0·9994	0·9997	0·9998	0·9999
4	0·9336	0·9577	0·9736	0·9838	0·9902	0·9942	0·9966	0·9980	0·9989	0·9994	0·9996
5	0·8803	0·9189	0·9463	0·9650	0·9776	0·9859	0·9913	0·9947	0·9968	0·9981	0·9989
6	0·8106	0·8647	0·9054	0·9352	0·9564	0·9712	0·9812	0·9880	0·9924	0·9952	0·9971
7	0·7282	0·7964	0·8508	0·8930	0·9247	0·9480	0·9646	0·9763	0·9844	0·9898	0·9934
8	0·6379	0·7171	0·7839	0·8384	0·8815	0·9147	0·9396	0·9580	0·9712	0·9805	0·9870
9	0·5453	0·6311	0·7075	0·7729	0·8271	0·8709	0·9052	0·9316	0·9514	0·9659	0·9765
10	0·4551	0·5431	0·6252	0·6990	0·7630	0·8169	0·8610	0·8963	0·9238	0·9449	0·9607

11	0·3713	0·4572	0·5412	0·6201	0·6915	0·7542	0·8076	0·8519	0·8878	0·9164	0·9385
12	0·2965	0·3769	0·4591	0·5396	0·6155	0·6848	0·7462	0·7990	0·8434	0·8799	0·9092
13	0·2320	0·3044	0·3818	0·4607	0·5381	0·6114	0·6788	0·7389	0·7911	0·8355	0·8723
14	0·1780	0·2413	0·3116	0·3861	0·4621	0·5368	0·6078	0·6733	0·7322	0·7838	0·8280
15	0·1342	0·1878	0·2497	0·3180	0·3900	0·4634	0·5356	0·6044	0·6683	0·7260	0·7770
16	0·0994	0·1436	0·1967	0·2574	0·3237	0·3936	0·4646	0·5345	0·6014	0·6637	0·7203
17	0·0725	0·1081	0·1524	0·2049	0·2643	0·3290	0·3968	0·4657	0·5335	0·5986	0·6594
18	0·0521	0·0801	0·1163	0·1606	0·2125	0·2708	0·3338	0·3997	0·4666	0·5326	0·5960
19	0·0369	0·0585	0·0874	0·1240	0·1683	0·2196	0·2767	0·3382	0·4024	0·4675	0·5318
20	0·0258	0·0422	0·0648	0·0944	0·1314	0·1755	0·2261	0·2822	0·3423	0·4049	0·4684

Table 2(a). Derivative of reduced temperature function. $T_2(\eta, \xi)_\epsilon - T_{w1}(\eta, \xi)$

η	$\xi = 0$	1	2	3	4	5	6	7	8	9	10
0	1·0000	0·3679	0·1353	0·0498	0·0183	0·0067	0·0025	0·0009	0·0003	0·0001	0·0000
1	0·3679	0·3087	0·2118	0·1311	0·0761	0·0422	0·0226	0·0118	0·0061	0·0030	0·0015
2	0·1353	0·2118	0·2072	0·1678	0·1220	0·0825	0·0529	0·0326	0·0194	0·0112	0·0064
3	0·0498	0·1311	0·1678	0·1668	0·1439	0·1132	0·0832	0·0580	0·0388	0·0251	0·0158
4	0·0183	0·0761	0·1220	0·1439	0·1435	0·1281	0·1056	0·0818	0·0604	0·0428	0·0293
5	0·0067	0·0422	0·0825	0·1132	0·1281	0·1279	0·1166	0·0992	0·0798	0·0614	0·0454
6	0·0025	0·0226	0·0529	0·0832	0·1056	0·1166	0·1165	0·1077	0·0937	0·0775	0·0615
7	0·0009	0·0118	0·0326	0·0580	0·0818	0·0992	0·1077	0·1076	0·1006	0·0891	0·0753
8	0·0003	0·0061	0·0194	0·0388	0·0604	0·0798	0·0937	0·1006	0·1006	0·0948	0·0850
9	0·0001	0·0030	0·0112	0·0251	0·0428	0·0614	0·0775	0·0891	0·0948	0·0947	0·0898
10	0·0000	0·0015	0·0064	0·0158	0·0293	0·0454	0·0615	0·0753	0·0850	0·0898	0·0898
11	0·0000	0·0007	0·0035	0·0096	0·0195	0·0325	0·0470	0·0612	0·0731	0·0814	0·0856
12	0·0000	0·0004	0·0019	0·0058	0·0127	0·0226	0·0348	0·0480	0·0606	0·0710	0·0783
13	0·0000	0·0002	0·0010	0·0034	0·0080	0·0153	0·0251	0·0366	0·0486	0·0598	0·0690
14	0·0000	0·0001	0·0006	0·0020	0·0050	0·0102	0·0177	0·0271	0·0379	0·0488	0·0589
15	0·0000	0·0000	0·0003	0·0011	0·0031	0·0066	0·0122	0·0197	0·0288	0·0388	0·0489
16	0·0000	0·0000	0·0002	0·0006	0·0018	0·0042	0·0082	0·0139	0·0214	0·0301	0·0395
17	0·0000	0·0000	0·0001	0·0004	0·0011	0·0027	0·0054	0·0097	0·0155	0·0228	0·0312
18	0·0000	0·0000	0·0000	0·0002	0·0006	0·0016	0·0035	0·0066	0·0111	0·0170	0·0241
19	0·0000	0·0000	0·0000	0·0001	0·0004	0·0010	0·0023	0·0044	0·0078	0·0124	0·0182
20	0·0000	0·0000	0·0000	0·0001	0·0002	0·0006	0·0014	0·0029	0·0053	0·0089	0·0135

Table 2(a)—contd.

η	$\xi = 10$	11	12	13	14	15	16	17	18	19	20
0	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000
1	0·0015	0·0007	0·0004	0·0002	0·0001	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000
2	0·0064	0·0035	0·0019	0·0010	0·0006	0·0003	0·0002	0·0001	0·0000	0·0000	0·0000
3	0·0158	0·0096	0·0058	0·0034	0·0020	0·0011	0·0006	0·0004	0·0002	0·0001	0·0001
4	0·0293	0·0195	0·0127	0·0080	0·0050	0·0031	0·0018	0·0011	0·0006	0·0004	0·0002
5	0·0454	0·0325	0·0226	0·0153	0·0102	0·0066	0·0042	0·0027	0·0016	0·0010	0·0006
6	0·0615	0·0470	0·0348	0·0251	0·0177	0·0122	0·0082	0·0054	0·0035	0·0023	0·0014
7	0·0753	0·0612	0·0480	0·0366	0·0271	0·0197	0·0139	0·0097	0·0066	0·0044	0·0029
8	0·0850	0·0731	0·0606	0·0486	0·0379	0·0288	0·0214	0·0155	0·0111	0·0078	0·0053
9	0·0898	0·0814	0·0710	0·0598	0·0488	0·0388	0·0301	0·0228	0·0170	0·0124	0·0089
10	0·0898	0·0856	0·0783	0·0690	0·0589	0·0489	0·0395	0·0312	0·0241	0·0182	0·0135
11	0·0856	0·0856	0·0819	0·0755	0·0672	0·0580	0·0488	0·0400	0·0320	0·0251	0·0194
12	0·0783	0·0819	0·0819	0·0787	0·0729	0·0655	0·0571	0·0486	0·0403	0·0327	0·0260
13	0·0690	0·0755	0·0787	0·0786	0·0758	0·0706	0·0639	0·0562	0·0483	0·0405	0·0333
14	0·0589	0·0672	0·0729	0·0758	0·0758	0·0732	0·0685	0·0624	0·0553	0·0479	0·0406
15	0·0489	0·0580	0·0655	0·0706	0·0732	0·0732	0·0708	0·0666	0·0609	0·0544	0·0475
16	0·0395	0·0488	0·0571	0·0639	0·0685	0·0708	0·0708	0·0687	0·0648	0·0596	0·0536
17	0·0312	0·0400	0·0486	0·0562	0·0624	0·0666	0·0687	0·0687	0·0667	0·0632	0·0584
18	0·0241	0·0320	0·0403	0·0483	0·0553	0·0609	0·0648	0·0667	0·0667	0·0650	0·0616
19	0·0182	0·0251	0·0327	0·0405	0·0479	0·0544	0·0596	0·0632	0·0650	0·0649	0·0633
20	0·0135	0·0194	0·0260	0·0333	0·0406	0·0475	0·0536	0·0584	0·0616	0·0633	0·0633

Table 2(b). Derivative of reduced temperature function. $-\ T_1(\eta, \xi),\ T_{w2}(\eta_w, \xi_w)$

ξ, η_w	$\eta, \xi_w = 0$	1	2	3	4	5	6	7	8	9	10
0	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000
1	0·3679	0·2155	0·1192	0·0635	0·0328	0·0166	0·0082	0·0040	0·0019	0·0009	0·0004
2	0·2707	0·2387	0·1789	0·1220	0·0782	0·0478	0·0282	0·0162	0·0091	0·0050	0·0027
3	0·1494	0·1907	0·1831	0·1521	0·1152	0·0818	0·0552	0·0358	0·0225	0·0138	0·0082
4	0·0733	0·1315	0·1565	0·1537	0·1342	0·1079	0·0816	0·0588	0·0408	0·0273	0·0178
5	0·0337	0·0831	0·1197	0·1363	0·1350	0·1213	0·1014	0·0802	0·0605	0·0440	0·0309
6	0·0149	0·0495	0·0848	0·1104	0·1225	0·1218	0·1115	0·0958	0·0782	0·0611	0·0461
7	0·0064	0·0282	0·0567	0·0836	0·1030	0·1123	0·1118	0·1037	0·0909	0·0760	0·0611
8	0·0027	0·0156	0·0363	0·0601	0·0816	0·0969	0·1042	0·1039	0·0974	0·0867	0·0739
9	0·0011	0·0083	0·0224	0·0413	0·0615	0·0792	0·0917	0·0978	0·0975	0·0921	0·0829
10	0·0004	0·0044	0·0134	0·0274	0·0445	0·0619	0·0768	0·0872	0·0923	0·0922	0·0875

11	0·0002	0·0022	0·0078	0·0177	0·0311	0·0466	0·0617	0·0745	0·0834	0·0877	0·0876
12	0·0001	0·0011	0·0045	0·0111	0·0212	0·0339	0·0478	0·0612	0·0723	0·0800	0·0838
13	0·0000	0·0006	0·0025	0·0068	0·0140	0·0240	0·0359	0·0486	0·0604	0·0702	0·0770
14	0·0000	0·0003	0·0014	0·0041	0·0090	0·0165	0·0262	0·0374	0·0489	0·0596	0·0683
15	0·0000	0·0001	0·0008	0·0024	0·0057	0·0111	0·0187	0·0281	0·0385	0·0490	0·0586
16	0·0000	0·0001	0·0004	0·0014	0·0036	0·0074	0·0130	0·0206	0·0296	0·0393	0·0490
17	0·0000	0·0000	0·0002	0·0008	0·0022	0·0048	0·0089	0·0147	0·0222	0·0307	0·0399
18	0·0000	0·0000	0·0001	0·0004	0·0013	0·0030	0·0060	0·0104	0·0163	0·0235	0·0317
19	0·0000	0·0000	0·0001	0·0002	0·0008	0·0019	0·0039	0·0072	0·0117	0·0176	0·0247
20	0·0000	0·0000	0·0000	0·0001	0·0005	0·0012	0·0026	0·0048	0·0083	0·0130	0·0188

ξ, η_w	$\eta, \xi_w = 10$	11	12	13	14	15	16	17	18	19	20
0	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000
1	0·0004	0·0002	0·0001	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000
2	0·0027	0·0014	0·0008	0·0004	0·0002	0·0001	0·0000	0·0000	0·0000	0·0000	0·0000
3	0·0082	0·0048	0·0028	0·0016	0·0009	0·0005	0·0003	0·0001	0·0001	0·0000	0·0000
4	0·0178	0·0113	0·0070	0·0043	0·0026	0·0015	0·0009	0·0005	0·0003	0·0002	0·0001
5	0·0309	0·0212	0·0141	0·0092	0·0059	0·0037	0·0023	0·0014	0·0008	0·0005	0·0003
6	0·0461	0·0336	0·0239	0·0166	0·0112	0·0075	0·0049	0·0031	0·0020	0·0012	0·0008
7	0·0611	0·0474	0·0357	0·0261	0·0187	0·0131	0·0090	0·0061	0·0040	0·0026	0·0017
8	0·0739	0·0606	0·0482	0·0372	0·0280	0·0205	0·0148	0·0104	0·0072	0·0049	0·0033
9	0·0829	0·0718	0·0600	0·0486	0·0383	0·0294	0·0221	0·0163	0·0118	0·0083	0·0058
10	0·0875	0·0796	0·0698	0·0592	0·0488	0·0391	0·0306	0·0234	0·0176	0·0130	0·0094
11	0·0876	0·0836	0·0767	0·0679	0·0583	0·0487	0·0397	0·0316	0·0246	0·0188	0·0141
12	0·0838	0·0837	0·0802	0·0740	0·0662	0·0575	0·0486	0·0401	0·0323	0·0256	0·0198
13	0·0770	0·0803	0·0802	0·0771	0·0716	0·0645	0·0566	0·0483	0·0403	0·0330	0·0264
14	0·0683	0·0743	0·0772	0·0772	0·0744	0·0694	0·0630	0·0557	0·0480	0·0405	0·0334
15	0·0586	0·0664	0·0718	0·0745	0·0744	0·0719	0·0675	0·0616	0·0548	0·0476	0·0406
16	0·0490	0·0577	0·0648	0·0696	0·0720	0·0720	0·0697	0·0656	0·0602	0·0539	0·0472
17	0·0399	0·0488	0·0568	0·0632	0·0676	0·0698	0·0697	0·0677	0·0589	0·0639	0·0531
18	0·0317	0·0403	0·0485	0·0559	0·0617	0·0657	0·0677	0·0677	0·0658	0·0624	0·0577
19	0·0247	0·0325	0·0405	0·0482	0·0550	0·0603	0·0640	0·0658	0·0658	0·0641	0·0609
20	0·0188	0·0256	0·0331	0·0406	0·0478	0·0541	0·0590	0·0624	0·0641	0·0641	0·0625

Table 3(a). Integral of reduced temperature function $U(\eta, \xi)$, $V_w(\eta_w, \xi_w)$

η, ξ_w	$\xi, \eta_w = 0$	1	2	3	4	5	6	7	8	9	10
0	0·0000	0·3679	1·135	2·050	3·018	4·007	5·002	6·001	7·000	8·000	9·000
1	0·0000	0·1778	0·6616	1·359	2·189	3·098	4·049	5·025	6·012	7·006	8·003
2	0·0000	0·0850	0·3748	0·8689	1·530	2·314	3·182	4·103	5·057	6·031	7·017
3	0·0000	0·0401	0·2074	0·5392	1·036	1·676	2·429	3·266	4·162	5·096	6·056
4	0·0000	0·0188	0·1125	0·3262	0·6818	1·180	1·805	2·536	3·349	4·222	5·139
5	0·0000	0·0087	0·0600	0·1930	0·4377	0·8092	1·308	1·922	2·636	3·429	4·284
6	0·0000	0·0040	0·0316	0·1120	0·2748	0·5423	0·9254	1·424	2·030	2·730	3·507
7	0·0000	0·0018	0·0164	0·0639	0·1692	0·3558	0·6408	1·033	1·532	2·131	2·819
8	0·0000	0·0008	0·0084	0·0359	0·1023	0·2290	0·4350	0·7341	1·133	1·632	2·226
9	0·0000	0·0004	0·0043	0·0199	0·0609	0·1448	0·2899	0·5121	0·8228	1·228	1·727
10	0·0000	0·0002	0·0021	0·0109	0·0357	0·0901	0·1899	0·3510	0·5870	0·9076	1·318
11	0·0000	0·0001	0·0011	0·0059	0·0207	0·0553	0·1225	0·2367	0·4119	0·6598	0·9888
12	0·0000	0·0000	0·0005	0·0032	0·0118	0·0334	0·0778	0·1572	0·2845	0·4722	0·7306
13	0·0000	0·0000	0·0003	0·0017	0·0067	0·0200	0·0488	0·1029	0·1936	0·3329	0·5318
14	0·0000	0·0000	0·0001	0·0009	0·0037	0·0118	0·0302	0·0665	0·1300	0·2314	0·3816
15	0·0000	0·0000	0·0001	0·0005	0·0021	0·0069	0·0185	0·0424	0·0861	0·1587	0·2701
16	0·0000	0·0000	0·0000	0·0002	0·0011	0·0040	0·0112	0·0267	0·0564	0·1075	0·1887
17	0·0000	0·0000	0·0000	0·0001	0·0006	0·0023	0·0067	0·0167	0·0364	0·0719	0·1303
18	0·0000	0·0000	0·0000	0·0001	0·0003	0·0013	0·0040	0·0103	0·0233	0·0476	0·0888
19	0·0000	0·0000	0·0000	0·0000	0·0002	0·0007	0·0023	0·0063	0·0148	0·0311	0·0599
20	0·0000	0·0000	0·0000	0·0000	0·0001	0·0004	0·0014	0·0038	0·0092	0·0202	0·0400

η, ξ_w	$\xi, \eta_w = 10$	11	12	13	14	15	16	17	18	19	20
0	9·000	10·00	11·00	12·00	13·00	14·00	15·00	16·00	17·00	18·00	19·00
1	8·003	9·001	10·00	11·00	12·00	13·00	14·00	15·00	16·00	17·00	18·00
2	7·017	8·009	9·005	10·00	11·00	12·00	13·00	14·00	15·00	16·00	17·00
3	6·056	7·032	8·018	9·010	10·01	11·00	12·00	13·00	14·00	15·00	16·00
4	5·139	6·086	7·052	8·031	9·018	10·01	11·01	12·00	13·00	14·00	15·00
5	4·284	5·185	6·118	7·075	8·046	9·028	10·02	11·01	12·01	13·00	14·00
6	3·507	4·346	5·232	6·153	7·109	8·064	9·040	10·03	11·02	12·01	13·01
7	2·819	3·583	4·407	5·280	6·190	7·127	8·084	9·054	10·03	11·02	12·01
8	2·226	2·905	3·656	4·468	5·329	6·228	7·156	8·105	9·070	10·05	11·03
9	1·727	2·316	2·986	3·727	4·528	5·378	6·267	7·186	8·128	9·087	10·06
10	1·318	1·817	2·402	3·065	3·797	4·587	5·427	6·306	7·217	8·152	9·105

11	0·9888	1·403	1·903	2·484	3·140	3·864	4·646	5·476	6·347	7·249	8·177
12	0·7306	1·067	1·485	1·985	2·563	3·213	3·930	4·703	5·525	6·387	7·282
13	0·5318	0·7995	1·142	1·564	2·063	2·638	3·284	3·993	4·759	5·573	6·428
14	0·3816	0·5906	0·8665	1·215	1·639	2·139	2.712	3·353	4·056	4·815	5·621
15	0·2701	0·4303	0·6484	0·9318	1·286	1·712	2·212	2·782	3·419	4·117	4·869
16	0·1887	0·3095	0·4790	0·7055	0·9956	1·354	1·783	2·283	2·851	3·484	4·177
17	0·1303	0·2198	0·3494	0·5274	0·7616	1·058	1·421	1·852	2·352	2·918	3·547
18	0·0888	0·1543	0·2518	0·3896	0·5756	0·8168	1·119	1·485	1·919	2·418	2·983
19	0·0599	0·1071	0·1794	0·2845	0·4300	0·6234	0·8711	1·178	1·548	1·983	2·483
20	0·0400	0·0735	0·1264	0·2054	0·3177	0·4706	0·6709	0·9246	1·237	1·610	2·047

Table 3(b). Integral of reduced temperature function. $U_w(\eta_w, \xi_w)$, $V(\eta, \xi)$

ξ, η_w	$\eta, \xi_w = 0$	1	2	3	4	5	6	7	8	9	10
0	0·0000	1·000	2·000	3·000	4·000	5·000	6·000	7·000	8·000	9·000	10·00
1	0·0000	0·5235	1·267	2·134	3·066	4·032	5·015	6·007	7·003	8·002	9·001
2	0·0000	0·2673	0·7712	1·454	2·260	3·146	4·080	5·043	6·023	7·012	8·006
3	0·0000	0·1338	0·4542	0·9559	1·609	2·378	3·229	4·136	5·079	6·046	7·026
4	0·0000	0·0659	0·2605	0·6091	1·110	1·745	2·487	3·312	4·195	5·120	6·073
5	0·0000	0·0320	0·1460	0·3779	0·7446	1·245	1·867	2·589	3·392	4·256	5·164
6	0·0000	0·0154	0·0803	0·2291	0·4872	0·8667	1·367	1·979	2·686	3·471	4·318
7	0·0000	0·0073	0·0434	0·1360	0·3116	0·5894	0·9786	1·479	2·083	2·777	3·548
8	0·0000	0·0034	0·0231	0·0793	0·1954	0·3925	0·6858	1·083	1·583	2·180	2·864
9	0·0000	0·0016	0·0121	0·0455	0·1203	0·2564	0·4712	0·7771	1·180	1·681	2·272
10	0·0000	0·0007	0·0063	0·0257	0·0728	0·1645	0·3178	0·5476	0·8641	1·272	1·773
11	0·0000	0·0003	0·0032	0·0143	0·0434	0·1038	0·2108	0·3792	0·6217	0·9473	1·360
12	0·0000	0·0002	0·0017	0·0079	0·0255	0·0646	0·1376	0·2583	0·4400	0·6937	1·027
13	0·0000	0·0001	0·0008	0·0043	0·0148	0·0396	0·0885	0·1733	0·3067	0·5002	0·7637
14	0·0000	0·0000	0·0004	0·0023	0·0085	0·0240	0·0561	0·1146	0·2106	0·3554	0·5596
15	0·0000	0·0000	0·0002	0·0012	0·0048	0·0143	0·0352	0·0748	0·1426	0·2490	0·4043
16	0·0000	0·0000	0·0001	0·0007	0·0027	0·0085	0·0218	0·0482	0·0954	0·1721	0·2882
17	0·0000	0·0000	0·0000	0·0004	0·0015	0·0050	0·0133	0·0307	0·0630	0·1175	0·2028
18	0·0000	0·0000	0·0000	0·0002	0·0008	0·0029	0·0080	0·0193	0·0411	0·0792	0·1410
19	0·0000	0·0000	0·0000	0·0001	0·0005	0·0016	0·0048	0·0120	0·0265	0·0528	0·0968
20	0·0000	0·0000	0·0000	0·0000	0·0002	0·0009	0·0029	0·0074	0·0169	0·0348	0·0658

Table 3(b)—contd.

ξ, η_w	$\eta, \xi_w = 10$	11	12	13	14	15	16	17	18	19	20
0	10·00	11·00	12·00	13·00	14·00	15·00	16·00	17·00	18·00	19·00	20·00
1	9·001	10·00	11·00	12·00	13·00	14·00	15·00	16·00	17·00	18·00	19·00
2	8·006	9·003	10·00	11·00	12·00	13·00	14·00	15·00	16·00	17·00	18·00
3	7·026	8·014	9·008	10·00	11·00	12·00	13·00	14·00	15·00	16·00	17·00
4	6·073	7·043	8·026	9·015	10·01	11·00	12·00	13·00	14·00	15·00	16·00
5	5·164	6·104	7·065	8·040	9·024	10·01	11·01	12·00	13·00	14·00	15·00
6	4·318	5·211	6·138	7·088	8·056	9·035	10·02	11·01	12·01	13·00	14·00
7	3·548	4·379	5·258	6·173	7·115	8·075	9·048	10·03	11·02	12·01	13·01
8	2·864	3·622	4·440	5·307	6·211	7·143	8·095	9·063	10·04	11·03	12·02
9	2·272	2·947	3·694	4·500	5·355	6·249	7·172	8·117	9·079	10·05	11·03
10	1·773	2·360	3·027	3·764	4·560	5·404	6·288	7·203	8·141	9·097	10·07
11	1·360	1·860	2·444	3·104	3·832	4·618	5·453	6·328	7·234	8·166	9·116
12	1·027	1·444	1·944	2·524	3·178	3·898	4·676	5·502	6·368	7·267	8·191
13	0·7637	1·104	1·524	2·024	2·601	3·250	3·963	4·732	5·550	6·409	7·300
14	0·5596	0·8318	1·178	1·601	2·101	2·676	3·319	4·026	4·788	5·598	6·449
15	0·4043	0·6181	0·8982	1·250	1·676	2·176	2·748	3·387	4·088	4·843	5·646
16	0·2882	0·4531	0·6757	0·9628	1·319	1·748	2·248	2·817	3·453	4·148	4·897
17	0·2028	0·3279	0·5018	0·7324	1·026	1·387	1·817	2·317	2·885	3·517	4·207
18	0·1410	0·2344	0·3681	0·5502	0·7881	1·088	1·453	1·885	2·385	2·951	3·579
19	0·0968	0·1656	0·2668	0·4085	0·5983	0·8430	1·148	1·517	1·951	2·451	3·015
20	0·0658	0·1156	0·1912	0·2998	0·4491	0·6560	0·8970	1·207	1·579	2·015	2·515

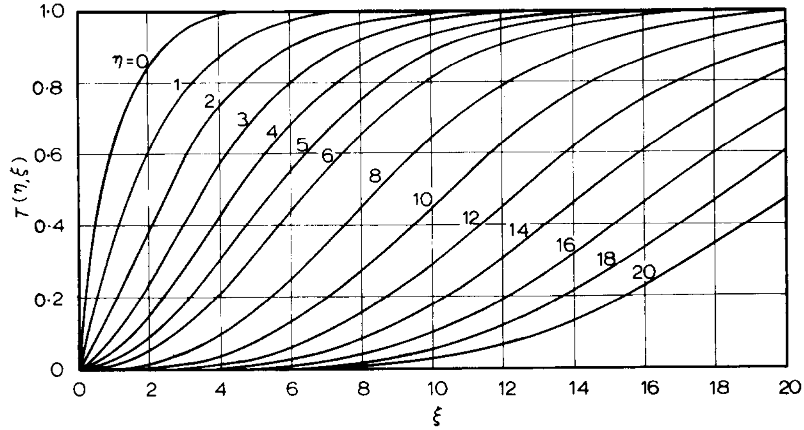

FIG. 1(a). Reduced fluid temperature function.

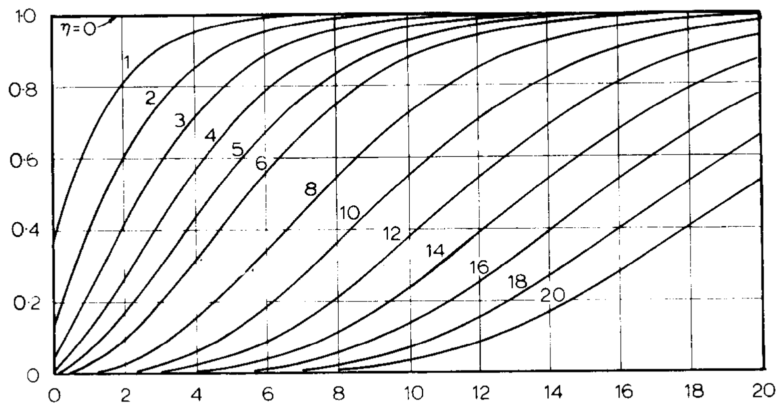

FIG. 1(b). Reduced wall temperature function.

functions are related in a very simple way [9]. That is, one of the functions is the complement of the other when the arguments are interchanged, i.e.

$$T(x, y) = 1 - T_w(y, x).$$

For this reason, a plot or table of one of these functions is sufficient for calculating both fluid and wall temperatures. This fact has permitted the conservation of space through the presentation of both T and T_w in a single, combined table (see Table 1). The subscripts in the row and column headings indicate how to enter the table.

Arbitrary initial condition

As a result of the linear, partial differential equations which govern the temperatures, one can superimpose the solutions to a number of simple problems in order to obtain the solution to a more difficult one. As an illustration, solutions to two simple cases will be given, both of which are found from the $T(\eta, \xi)$, $T_w(\eta, \xi)$ solutions presented above. These two solutions will then be added resulting in the solution to a third.

Consider a given matrix with fixed conditions of: C, c_p, h, M, $\dot{m}$ and P.

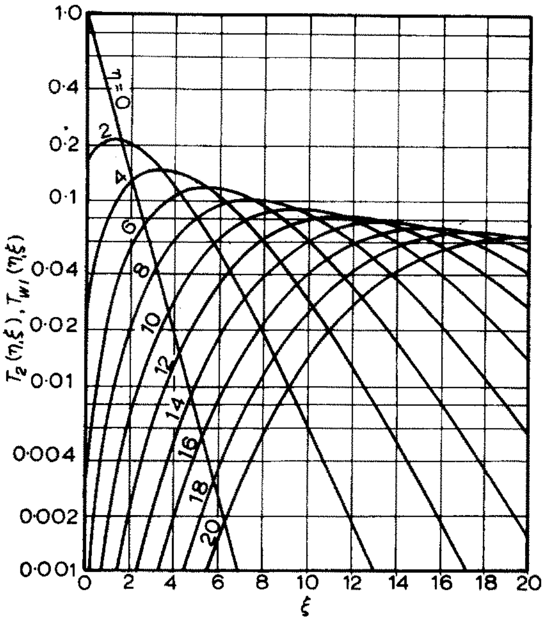

FIG. 2(a). Derivative of reduced temperature function.

Case (a)

Fluid enters at temperature u_o, constant. Initially wall at temperature v_o, uniform. See Fig. 5 (a).

Solution:

$$t_a(\eta, \xi) = u_o + (v_o - u_o)T(\eta, \xi)$$

and

$$t_{wa}(\eta, \xi) = u_o + (v_o - u_o)T_w(\eta, \xi).$$

Case (b)

Fluid enters at temperature zero, constant. Initially wall at temperature: zero,

$$0 \leqslant \zeta < \zeta_1$$

and

$$v_1, \zeta_1 \leqslant \zeta \leqslant \xi.$$

See Fig. 5 (b).

Solution:

$$t_b(\eta, \xi) = v_1 T(\eta, \xi - \zeta_1)$$

and

$$t_{wb}(\eta, \xi) = v_1 T_w(\eta, \xi - \zeta_1).$$

Case (c)

Fluid enters at temperature u_o, constant. Initially wall at temperature: v_o,

$$0 \leqslant \zeta < \zeta_1$$

and

$$v_o + v_1, \zeta_1 \leqslant \zeta \leqslant \xi.$$

See Fig. 5 (c).

Here, the boundary condition u and the initial condition v are equal to the sum of those of cases (a) and (b), respectively. Therefore, the solution is just equal to the sum of the solutions of cases (a) and (b).

Solution:

$$t_c(\eta, \xi) = u_o + (v_o - u_o)T(\eta, \xi) + v_1 T(\eta, \xi - \zeta_1)$$

and

$$t_{wc}(\eta, \xi) = u_o + (v_o - u_o)T_w(\eta, \xi) + v_1 T_w(\eta, \xi - \zeta_1).$$

The reader will observe that, in each case, the solution for t includes the function T, and the solution for t_w includes the function T_w. Except for this difference, the respective solutions are identical. All further derivations will follow this

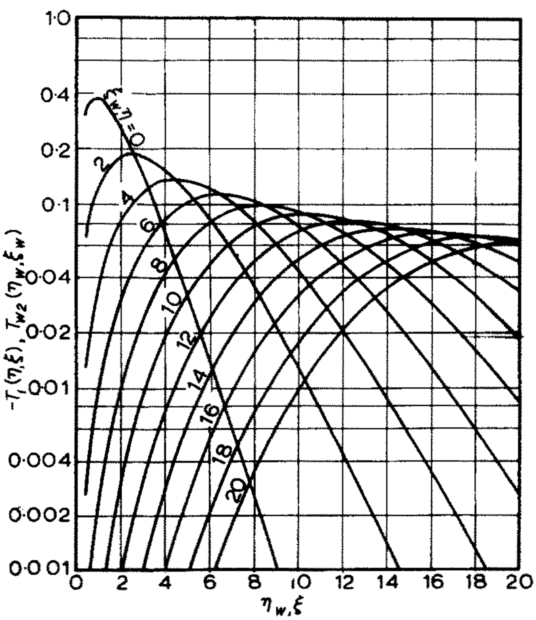

FIG. 2(b). Derivative of reduced temperature function.

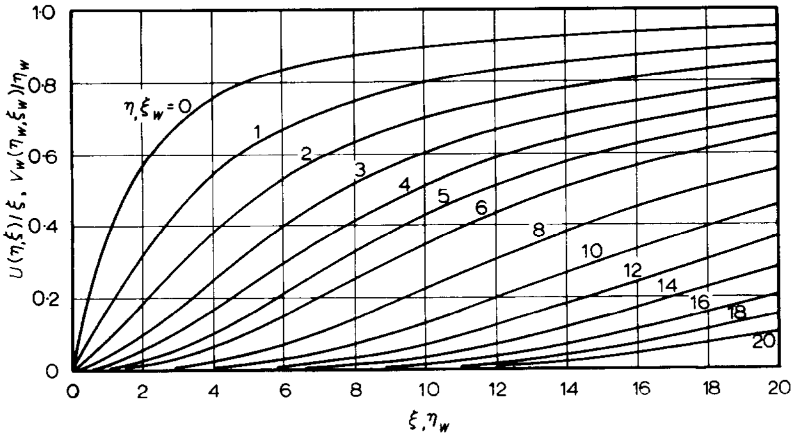

FIG. 3(a). Integral of reduced temperature function divided by limit of integration.

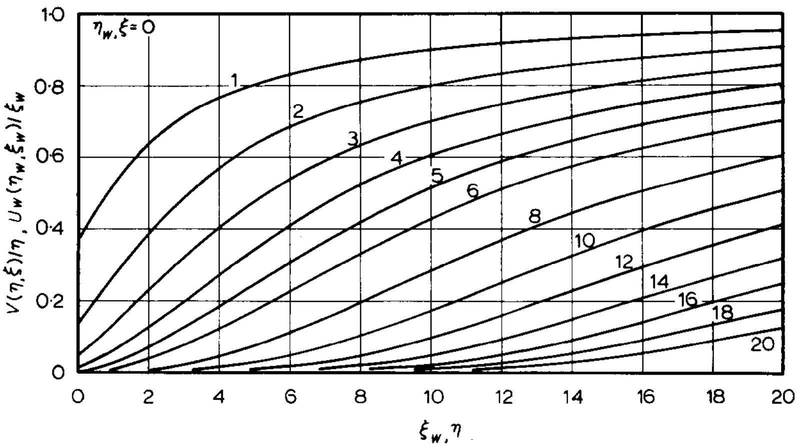

FIG. 3(b). Integral of reduced temperature function divided by limit of integration.

trend; therefore, only fluid temperatures will be treated in detail except where individual attention may be required.

The result of case (c) can be immediately extended to the situation where the initial distribution consists of any number of steps of height v_j located at $\zeta_j (j = 1, 2, 3, \ldots n)$. Then, for $\xi \geqslant \zeta_n$

$$t(\eta, \xi) = u_o + (v_o - u_o)T(\eta, \xi) + \sum_{j=1}^{n} v_j T(\eta, \xi - \zeta_j).$$

Letting the number of steps go to infinity, any continuous, initial temperature distribution can be accommodated through Duhamel's integral [10]. This result is

$$t(\eta, \xi) = u_o + (v_o - u_o)T(\eta, \xi) + \int_0^{\xi} v' T(\eta, \xi - \zeta)\, \mathrm{d}\zeta \quad (7)$$

where $v' = \mathrm{d}v/\mathrm{d}\zeta$.
Integration by parts yields

$$t(\eta, \xi) = u_o\{1 - T(\eta, \xi)\} + v_\xi T(\eta, 0) + \int_0^{\xi} v T_2(\eta, \xi - \zeta)\, \mathrm{d}\zeta \quad (8)$$

where $T_2(x, y) = \partial T(x, y)/\partial y$,
and v_ξ is the initial temperature at ξ.

This relationship is the working equation for the first method referred to in the abstract. The partial derivative T_2 is presented in Fig. 2(a) and is also tabulated in Table 2(a). The detailed account of its calculation is given in the Appendix. Having the initial temperature distribution and the tables or curves of T_2, one can readily calculate the desired temperature. The

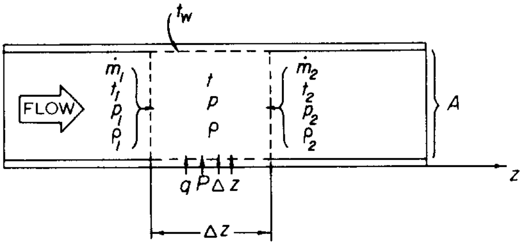

FIG. 4. Model.

integral may be evaluated by graphical or numerical means. The partial derivative T_{w2}, for use in making the analogous wall temperature calculation, is presented in Fig. 2(b) and Table 2(b).

The second method concerns equation (7). This is generally the more approximate method in that the initial temperature distribution is approximated by a small number of linear segments. Suppose, first, that we have an initial temperature distribution approximated by a single linear relationship such as $v = v_o + a\zeta$.

The derivative ($v' = a$) may now be placed in front of the integral sign and equation (7) becomes

$$t(\eta, \xi) = u_o + (v_o - u_o)T(\eta, \xi) + aU(\eta, \xi)$$

where the function*

$$U(\eta, \xi) = \int_0^\xi T(\eta, \xi - \zeta)\, d\zeta = \int_0^\xi T(\eta, \zeta)\, d\zeta$$

* $U(\eta, \xi)$ is just the solution for the case where $u = 0$ and $v = \zeta$. Details regarding its calculation are included in the Appendix.

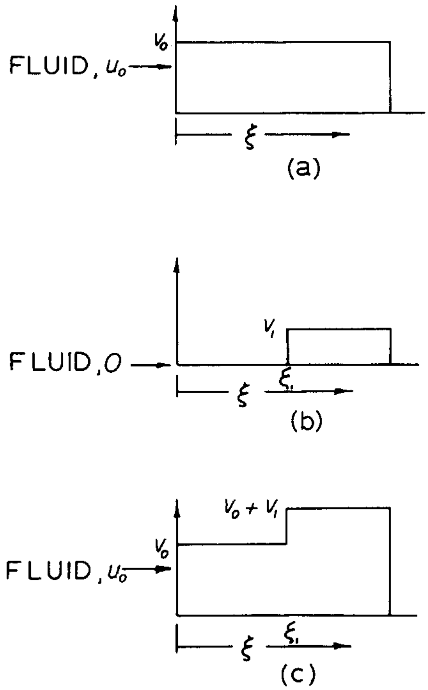

FIG. 5. Initial wall temperature distributions.

is presented in Table 3(a) and Fig. 3(a). The corresponding integral of the wall temperature function is presented in Table 3(b) and Fig. 3(b). In general, with n linear segments, each starting at ζ_{j-1} with slope $a_j (j = 1, 2, 3 \ldots n)$ the result is

$$t(\eta, \xi) = u_o + (v_o - u_o)T(\eta, \xi) + a_1 U(\eta, \xi) + \sum_{j=1}^{n-1} (a_{j+1} - a_j)U(\eta, \xi - \zeta_j) \quad (9)$$

the working equation of the second method.

How to compose a good approximation of the initial temperature distribution is a question of interest at this point. Equation (8) helps to shed some light on this subject. Observe that the weighting function $T_2(\eta, \xi - \zeta)$ causes the sensitivity of the solution to the initial distribution to vary with ζ. Observe also in Fig. 1 how the slope, T_2, of each curve is a maximum at about $\xi = \eta$. For this reason, any approximating distribution should be in closest agreement possible in the neighborhood of $\zeta = \xi - \eta$, provided $\eta < \xi$. It is suggested that three or four linear segments can effect the desired simplification for a curve having no more than one inflection point. A satisfactory criterion has been to have the approximate distribution agree with the actual distribution at the end points $\zeta = 0, \xi$, and also at $\zeta = \xi - \eta$. In the event that $\eta \geqslant \xi$, it is advantageous to have the approximate distribution agree most closely in the neighborhood of $\zeta = 0$.

Example. A sample problem will now be worked where the initial wall temperature is given as a continuous function of ζ and the fluid enters at the constant temperature $u_o = 100$°F. Both methods will be used to find the fluid temperature at $\eta = 6$, $\xi = 16$ for illustration. The initial temperature distribution v is plotted in Fig. 6 and tabulated in Table 4. Further calculations for the first method, equation (8), are also carried out in this table. Using Simpson's Rule,

$$\int_0^{\xi} vT_2(\eta, \xi - \zeta)\, d\zeta = 1511{\cdot}02.$$

Now, $T(6, 16) = 0{\cdot}9812$ (from Table 1)
$T(6, 0) = 0$.

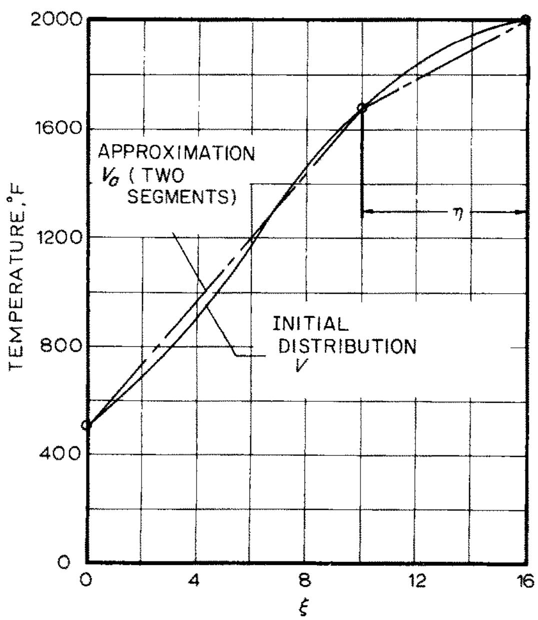

FIG. 6. Example initial wall temperature distribution.

Using equation (8)

$$t(6, 16) = 100\,\{1\text{–}0{\cdot}9812\} + 0 + 1511$$
$$1513\text{°F}.$$

The second method involves approximating the initial temperature distribution with several straight lines (two in the present case). This has

Table 4. Evaluation of integral in equation (8). $\eta = 6$

ζ	v(°F)	$\xi - \zeta$	$T_2(\eta, \xi - \zeta)$ [from Table 2(a)]	$vT_2(\eta, \xi - \zeta)$
0	500	16	0·0082	4·10
2	680	14	0·0177	12·04
4	910	12	0·0348	31·67
6	1190	10	0·0615	73·18
8	1462	8	0·0937	136·99
10	1675	6	0·1165	195·14
12	1838	4	0·1056	194·09
14	1940	2	0·0529	102·63
16	2000	0	0·0025	5·00

been done in Fig. 6. Values of ζ_j and a_j are found presently. The approximate distribution will be designated by v_a.

$$\zeta_1 = 10, \qquad \xi = 16$$
$$v_a(0) = 500, \quad v_a(\zeta_1) = 1675, \quad v_a(\xi) = 2000$$

$$a_1 = (1675\text{–}500)/10 = 117{\cdot}5,$$
$$a_2 = (2000\text{–}1675)/6 = 54{\cdot}17$$
$$v_o - u_o = 400.$$

Now, $T(6, 16) = 0{\cdot}9812$ (from Table 1)
$U(6, 16) = 9{\cdot}0405$ [from Table 3(a)]
$U(6, 6) = 0{\cdot}9254$.

Using equation (9)

$$t(6, 16) = 100 + 400(0{\cdot}9812) + 117{\cdot}5(9{\cdot}0405)$$
$$+ (54{\cdot}17 - 117{\cdot}5)\,0{\cdot}9254 = 1496\text{°F}.$$

It is seen that with this example, these two methods give answers which agree within about 1 per cent.

Arbitrary boundary condition

Following the same procedure as used for the arbitrary initial condition described above, it is possible to superimpose the solutions of several cases having simple boundary conditions so as to yield the solution to one having a more complicated boundary condition. In order to illustrate this, the solution of case (a) above and that of case (d), to be given subsequently, will be added to give the solution to case (e). Fixed conditions on the quantities C, c_p, h, M, $\dot{m}$, and p will be maintained.

Case (d)

Fluid enters at temperature:

$$\text{zero}, \quad 0 \leqslant \tau < \tau_1;$$

and

$$u_1, \quad \tau_1 \leqslant \tau \leqslant \eta.$$

Initially wall at temperature zero, uniform.

Solution:

$$t_d(\eta, \xi) = u_1 - u_1 T(\eta - \tau_1, \xi).$$

Case (e)

Fluid enters at temperature:

$$u_o, \quad 0 \leqslant \tau < \tau_1;$$

and

$$u_o + u_1, \quad \tau_1 \leqslant \tau \leqslant \eta.$$

Initially wall temperature v_o, uniform.

Solution:

$$t_e(\eta, \xi) = u_o + (v_o - u_o) T(\eta, \xi) + u_1 - u_1 T(\eta - \tau_1, \xi).$$

Generalizing this result to the case where the entering fluid temperature history consists of any number of steps of height u_j occurring at $\tau_j (j = 1, 2, 3, \ldots m)$ there results for $\eta \geqslant \tau_m$

$$t(\eta, \xi) = u_o + (v_o - u_o) T(\eta, \xi) + \sum_{j=1}^{m} \{u_j - u_j T(\eta - \tau_j, \xi)\}.$$

Again, letting the number of steps go to infinity, any continuous history of entering fluid temperature can be accommodated.

$$t(\eta, \xi) = u_o + (v_o - u_o) T(\eta, \xi) + \int_0^{\eta} \{u' - u' T(\eta - \tau, \xi)\} \, d\tau \tag{10}$$

where $u' = du/d\tau$.

Integrating by parts

$$t(\eta, \xi) = u_\eta \{1 - T(0, \xi)\} + v_o T(\eta, \xi) - \int_0^{\eta} u T_1(\eta - \tau, \xi) \, d\tau \tag{11}$$

where $T_1(x, y) = \partial T(x, y)/\partial x$

and u_η is the entering fluid temperature at dimensionless time η. This partial derivative is given in Fig. 2(b) and is tabulated in Table 2(b). With these values, one may readily evaluate the integral for any continuous entering fluid temperature history. Thus, the required temperature is found using this equation in the same manner as given above for equation (8).

Returning to equation (10), we have the method whereby the temperature can be calculated using an approximating entering fluid temperature history function, u_a. This function is made up of several linear segments analogous to that used with the second method given for the arbitrary initial temperature distribution problem. The derivative

$$du_a/d\tau = b_1, \quad 0 \leqslant \tau < \tau_1; \qquad b_2, \quad \tau_1 \leqslant \tau < \tau_2; \text{ etc.}$$

Equation (10) becomes

$$t(\eta, \xi) = u_o + (v_o - u_o) T(\eta, \xi) + b_1 V(\eta, \xi) + \sum_{j=1}^{m-1} (b_{j+1} - b_j) V(\eta - \tau_j, \xi) \tag{12}$$

where

$$V(\eta, \xi) = \eta - \int_0^{\eta} T(\tau, \xi) \, d\tau.$$

This function* is given in Fig. 3(b) and Table 3(b) with $U_w(\eta, \xi)$. Certain relationships involved with its evaluation are included in the Appendix.

Analogous recommendations regarding the construction of the approximating fluid temperature history are: three or four linear segments with agreement at end points $\tau = 0$, η and at the point (time) $\tau = \eta - \xi$, provided

* $V(\eta, \xi)$ is just the solution for the case where $u = \tau$ and $v = 0$.

$\eta > \xi$. When $\eta \leqslant \xi$ it is advantageous to approximate the function most closely in the neighborhood of $\tau = 0$.

Combination of both arbitrary conditions

The combined problem where both arbitrary conditions exist simultaneously is easily written down. The arbitrary initial wall temperature solution having zero entering fluid temperature is added to the arbitrary entering fluid temperature history solution having zero initial wall temperature. With the first method, using equations (8) and (11)

$$t(\eta, \xi) = v_\xi T(\eta, 0) + \int_0^\xi v T_2(\eta, \xi - \zeta)\, d\zeta + u_\eta\{1 - T(0, \xi)\} - \int_0^\eta u T_1(\eta - \tau, \xi)\, d\tau. \quad (13)$$

With the second method, using equations (9) and (12)

$$t(\eta, \xi) = u_o + (v_o - u_o) T(\eta, \xi) + a_1 U(\eta, \xi) + \sum_{j=1}^{n-1} (a_{j+1} - a_j) U(\eta, \xi - \zeta_j) + b_1 V(\eta, \xi) + \sum_{j=1}^{m-1} (b_{j+1} - b_j) V(\eta - \tau_j, \xi). \quad (14)$$

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APPENDIX

The equations

$$\frac{\partial T}{\partial \xi} = T_w - T \quad (5)$$

$$\frac{\partial T_w}{\partial \eta} = T - T_w \quad (6)$$

with the conditions

$$T = 0 \quad \text{at} \quad \xi = 0 \quad (15)$$

and

$$T_w = 1 \quad \text{at} \quad \eta = 0 \quad (16)$$

have been solved numerically. First, holding $\eta = 0$, equation (5) was solved for $T(0, \xi)$.

$$T(0, \xi) = 1 - \exp(-\xi). \quad (17)$$

Second, holding $\xi = 0$, equation (6) was solved for $T_w(\eta, 0)$.

$$T_w(\eta, 0) = \exp(-\eta). \quad (18)$$

Equations (15) through (18) were used as starting values for the step-by-step finite difference calculation described presently. Over an increment of length $\Delta\xi$ the fluid temperature changes by the increment $\Delta_\xi T$, according to equation (5).

$$\Delta_\xi T = \overline{(T_w - T)}_{\Delta\xi} \Delta\xi. \quad (19)$$

Similarly, according to equation (6)

$$\Delta_\eta T_w = \overline{(T - T_w)}_{\Delta\eta} \Delta\eta \quad (20)$$

The temperature differences are mean values which exist over the respective increments of the independent variables, $\Delta\xi$ and $\Delta\eta$. Using backward differences, let:

$$\overline{(T_w - T)}_{\Delta\xi} = \{T_w(\eta, \xi) + T_w(\eta, \xi - \Delta\xi) - T(\eta, \xi) - T(\eta, \xi - \Delta\xi)\}/2$$

$$\overline{(T - T_w)}_{\Delta\eta} = \{T(\eta, \xi) + T(\eta - \Delta\eta, \xi) - T_w(\eta, \xi) - T_w(\eta - \Delta\eta, \xi)\}/2$$

$$\Delta_\xi T = T(\eta, \xi) - T(\eta, \xi - \Delta\xi)$$

and

$$\Delta_\eta T_w = T_w(\eta, \xi) - T_w(\eta - \Delta\eta, \xi).$$

Substituting these definitions into equations (19) and (20) yields

$$T(\eta, \xi) = \frac{\Delta\xi}{2 + \Delta\xi}\{T_w(\eta, \xi) + T_w(\eta, \xi - \Delta\xi)\} + \frac{2 - \Delta\xi}{2 + \Delta\xi} T(\eta, \xi - \Delta\xi) \quad (21)$$

and

$$T_w(\eta, \xi) = \frac{\Delta\eta}{2 + \Delta\eta}\{T(\eta, \xi) + T(\eta - \Delta\eta, \xi)\} + \frac{2 - \Delta n}{2 + \Delta\eta} T_w(\eta - \Delta\eta, \xi). \quad (22)$$

Now, as they stand, equations (21) and (22) cannot be used to give explicit values of both $T(\eta, \xi)$ and $T_w(\eta, \xi)$; (note that one is needed to calculate the other). Iteration can be used at each point in order to cope with this difficulty. Another way is to solve the two equations simultaneously. Setting $\Delta\eta = \Delta\xi$, the result for $T(\eta, \xi)$ is

$$\left.\begin{aligned} T(\eta, \xi) = {} & \frac{4 - \Delta\xi^2}{4 + 4\Delta\xi} T(\eta, \xi - \Delta\xi) \\ & + \frac{2\Delta\xi + \Delta\xi^2}{4 + 4\Delta\xi} T_w(\eta, \xi - \Delta\xi) \\ & + \frac{2\Delta\xi - \Delta\xi^2}{4 + 4\Delta\xi} T_w(\eta - \Delta\eta, \xi) \\ & + \frac{\Delta\xi^2}{4 + 4\Delta\xi} T(\eta - \Delta\eta, \xi). \end{aligned}\right\} \quad (23)$$

For stability, the coefficients of all temperatures should be positive. This criterion is easily satisfied by $\Delta\eta, \Delta\xi \leqslant 2$. Through the course of the calculations, the increments have been varied. It was found that the fourth decimal place could be achieved by taking $\Delta\eta = \Delta\xi = \frac{1}{8}$. Using equations (23) and (22), in that order, the temperatures $T(\eta, \xi)$ and $T_w(\eta, \xi)$ have been worked out explicitly at each point. The results are those found in Table 1 and Fig. 1.

The exact solution of equations (5) and (6) with constant entering fluid temperature has been given by Nusselt* [2]. With a uniform initial wall temperature it reduces to

$$T(\eta, \xi) = \exp(-\eta)\int_0^\xi \exp(-\zeta) I_0[2\sqrt{(\eta\zeta)}]\,d\zeta \quad (24)$$

$$T_w(\eta, \xi) = \exp(-\eta)\left\{1 + (\sqrt{\eta}) \times \int_0^\xi \frac{\exp(-\zeta)}{\sqrt{(\zeta)}} I_1[2\sqrt{(\eta\zeta)}]\,d\zeta\right\}. \quad (25)$$

Using equation (24), $T(\eta, \xi)$ has been calculated for $\eta = 5$ and 10 at several points. The results are compared with the numerical solution in Table A1.

Table A1. *Comparison of exact and numerical solutions*

$\eta = 5$			$\eta = 10$		
ξ	T(exact)	T(num)†	ξ	T(exact)	T(num)‡
0·05	0·00037	0·00038	0·10	0·00001	0·00002
0·20	0·00192	0·00191	0·40	0·00007	0·00008
0·45	0·00609	0·00608	0·90	0·00044	0·00049
0·80	0·01570	0·01568	1·60	0·00213	0·00221
1·25	0·03520	0·03515	2·50	0·00833	0·00832
1·80	0·07032	0·07024	3·60	0·02657	0·02672
2·45	0·12678	0·12670	4·90	0·06993	0·07028
3·20	0·20813	0·20803	6·40	0·15373	0·15399
4·05	0·31343	0·31338	8·10	0·28547	0·28569
5·00	0·43608	0·43605	10·00	0·45494	0·45506

† Values interpolated from numerical results using a third degree polynomial, with $\Delta\xi$ spacings of $\frac{1}{8}$.

‡ Values interpolated from numerical results using a third degree polynomial, with $\Delta\xi$ spacings of $\frac{1}{4}$.

For use with equations (8), (11) and (13), the derivatives T_1, T_2, T_{w1}, and T_{w2} are needed. The governing partial differential equations readily give

$$T_2(\eta, \xi) = -T_{w1}(\eta, \xi) = T_w - T.$$

Values of these derivatives have been calculated

* Jakob [6] presents this solution, which provides for an arbitrary initial wall temperature distribution.

thusly during the computation of T and T_w. They are presented in Fig. 2(a) and Table 2(a).

The derivative T_{w2} has been computed using a five-point formula [11] during the computation of T and T_w. The size of increments used during computation was $\Delta\xi = \frac{1}{8}$. The formula used was

$$f'_0 = \frac{1}{12\Delta\xi}\{f_{-2} - 8f_{-1} + 8f_1 - f_2\}$$

where the subscripts indicate increments to one side or the other of ξ. The results are presented in Fig. 2(b) and Table 2(b). Obtaining this derivative at $\xi = 0$, however, was not possible with this formula. Rather than using an off-center formula here, it was calculated from the derivative of the exact solution, equation (25). That is

$$T_{w2}(\eta, 0) = \eta \exp(-\eta).$$

In the text it is stated that $T(x, y) = 1 - T_w(y, x)$. Partial differentiation of this relationship with respect to x gives

$$T_1(x, y) = -T_{w2}(y, x).$$

Therefore, T_1 is found from Table 2(b) also, but the arguments are interchanged.

The function

$$U(\eta, \xi) = \int_0^\xi T(\eta, \zeta)\,d\zeta$$

is just the solution where the fluid enters at temperature zero, and the initial wall temperature distribution is $v = \xi$. The initial temperature distribution in the fluid is

$$t(0, \xi) = \xi + \exp(-\xi) - 1,$$

therefore, the functions U and U_w have been evaluated by inserting these relationships for initial temperature distributions and calculating equations (23) and (22) precisely as done in obtaining T and T_w.

The function

$$V(\eta, \xi) = \eta - \int_0^\eta T(\tau, \xi)\,d\tau$$

for use with the arbitrary entering fluid temperature problem is found to be conveniently related to the function U_w. When the identity

$$T(\tau, \xi) = 1 - T_w(\xi, \tau)$$

is substituted into the above relationship, the result is

$$V(\eta, \xi) = U_w(\xi, \eta).$$

Similarly,

$$V_w(\eta, \xi) = U(\xi, \eta).$$

Therefore, these functions are to be found in Table 3 with U and U_w. The subscripted arguments of the table headings indicate how enter the table.

Résumé—Il est possible, au moyen de simples calculs à la main, de déterminer les températures du fluide et du solide à n'importe quel moment et à n'importe quel endroit d'un échangeur de chaleur à récupération. Les méthodes décrites s'appliquent aux cas où la température initiale varie arbitrairement avec la position longitudinale dans la matrice etoù la température du fluide à l'entrée varie avec le temps d'une façon arbitraire. La solution du problème avec une température initiale de matrice uniforme et une température constante du fluide à l'entrée a été publiée auparavant [1–4] et présentée ici sous forme de tableaux et de courbes pour les valeurs des paramètres η et ξ de 0 à 20. Les solutions pour le cas d'une distribution linéaire de la température initiale de matrice et une variation linéaire de la température du fluide à l'entrée sont également présentées sous forme de tableaux et de courbes dans la même gamme de paramètres. On a étendu, par superposition, ces résultats aux cas d'une température initiale de matrice arbitraire et d'une température de fluide à l'entrée arbitraire. Deux méthodes sont utiles pour l'obtention de résultats numériques. L'une consiste à calculer une intégrale de convolution qui implique la condition arbitraire. L'autre consiste à approcher la condition initiale (et/ou à la limite) arbitraire par un certain nombre de segments linéaires et à superposer les solutions tabulées.

Zusammenfassung—Mit Hilfe einfacher Handrechnungen ist es möglich, die Temperaturen von Flüssigkeit und Wand zu beliebigen Zeiten und an beliebigen Stellen für einen regenerativen Wärmeübertrager zu bestimmen. Die beschriebenen Methoden gelten für Fälle, in denen sich die Anfangstemperatur beliebig mit der Längsrichtung in der Matrize ändert und die Eintrittstemperatur der Flüssigkeit beliebig zeitveränderlich ist. Die Lösung des Problems mit einheitlicher Matrize für die Anfangstemperatur und konstante Eintrittstemperatur ist bereits früher veröffentlicht (1–4] und wird hier in Form von Tabellen und Kurven für Werte der Parameter η und ξ von 0 bis 20 wiedergegeben. Die Lösungen für eine lineare Matrize der Anfangstemperatur und lineare Änderung der Eintrittstemperatur sind ebenfalls in Form von Tabellen und Kurven im gleichen Parameterbereich angegeben. Durch Überlagerung werden diese Ergebnisse auf die Fälle mit beliebiger Matrize für die Anfangstemperatur erweitert. Von zwei Methoden ist jede nützlich zur Ermittlung numerischer Ergebnisse. Nach der einen Methode ist ein Linienintegral unter Einschluss einer beliebigen Bedingung auszuwerten; nach der anderen eine beliebige Anfangs- (und/oder Rand-) bedingung durch eine Reihe linearer Segmente anzunähern und tabellierten Lösungen zu überlagern.

Аннотация—С помощью простых расчетов можно определить температуры жидкости и твердого тела в любое время и в любом месте в регенеративном теплообменнике. Описанные методы применяются к случаям, где начальная температура произвольно изменяется с изменением продольной координаты в матрице, а температура жидкости на входе — со временем. Решение задачи об определении начальной температуры, записанной в виде однородной матрицы при постоянной температуре жидкости на входе, было ранее опубликовано в [1–4]. В данной работе это решение представлено в виде таблиц и кривых в интервале параметров η и ξ от 0 до 20. Также в виде таблиц и кривых в том же диапазоне параметров приводятся решения для температуры, записанной в виде линейной начальной матрицы, и для температур жидкости на входе, изменяющихся по линейному закону. Пользуясь суперпозицией, эти результаты можно применить к случаю произвольных матриц начальной температуры и жидкости на входе. Каждый из двух методов можно использовать для получения численных результатов. Первый метод позволяет определить интеграл свертки, который учитывает произвольное условие; другой допускает аппроксимацию произвольного начального (или граничного) условия с помощью нескольких линейных сегментов и комбинацию протабулированных решений.